

On some inequalities for submanifolds of Bochner Kaehler manifolds

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Abstract

B. Y. Chen established sharp inequalities between certain Riemannian invariants and the squared mean curvature for submanifolds in real space form as well as in complex space form. In this paper we generalize Chen inequalities for submanifolds of Bochner Kaehler manifolds. Moreover, we consider CR-warped product submanifolds of Bochner Kaehler manifold and establish an inequality for scalar curvature.

Keywords: Bochner Kaehler manifold, CR-warped product, slant submanifolds, Einstein manifold, Chen inequality.

2010 Mathematics Subject Classification: 53C15, 53C25, 53C40.

1. Introduction

In [6], B. Y. Chen established sharp inequality for a submanifold in a real space form involving intrinsic invariants of the submanifolds and squared mean curvature, the main extrinsic invariant and in [3], B. Y. Chen obtained the same inequality for complex space form. After that many research articles [7, 8, 9] have been published by different authors for different submanifolds and ambient spaces in complex as well as in contact version. In this article we obtain these inequalities for submanifolds in Bochner Kaehler manifold.

In [2] Bishop and O'Neil initiated the theory of warped product submanifold as a generalization of pseudo-Riemannian product manifold. In [5] Chen introduced the notion of CR-warped products. In This paper we study the CR-warped product submanifolds of Bochner Kaehler manifolds.

2. Preliminaries

Let \mathcal{W} be a n -dimensional submanifold of a Bochner Kaehler manifold $\overline{\mathcal{W}}$ of dimension $2m$. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on \mathcal{W} and $\overline{\mathcal{W}}$ respectively. Let J be the complex

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structure on $\overline{\mathcal{W}}$. Then the Gauss and Weingarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + \omega(X, Y), \quad (1)$$

$$\overline{\nabla}_X V = -B_V X + \nabla_X^\perp Y, \quad (2)$$

for all X, Y tangent to \mathcal{W} and vector field V normal to \mathcal{W} . Where ω , ∇_X^\perp , B_V denotes the second fundamental form, normal connection and the shape operator respectively. The second fundamental form and the shape operator are related by

$$g(\omega(X, Y), V) = g(B_V X, Y). \quad (3)$$

Let R be the curvature tensor of \mathcal{W} , Then the Gauss equation is given by [6]

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(\omega(X, W), \omega(Y, Z)) - g(\omega(X, Z), \omega(Y, W))$$

for any vector fields X, Y, Z, W tangent to \mathcal{W} .

The curvature tensor of a Bochner Kaehler manifold $\overline{\mathcal{W}}$ is given by [10]

$$\begin{aligned} \overline{R}(X, Y, Z, W) = & L(Y, Z)g(X, W) - L(X, Z)g(Y, W) + L(X, W)g(Y, Z) \\ & - L(Y, W)g(X, Z) + M(X, W)g(JX, W) - M(X, Z)g(JY, W) \\ & + M(X, W)g(JY, Z) - M(Y, W)g(JX, Z) \\ & - 2M(X, Y)g(JZ, W) - 2M(Z, W)g(JX, Y) \end{aligned} \quad (4)$$

where

$$L(Y, Z) = \frac{1}{2n+4} Ric(Y, Z) - \frac{\rho}{2(2n+2)(2n+4)} g(Y, Z), \quad (5)$$

$$M(Y, Z) = -L(Y, JZ), \quad (6)$$

$$L(Y, Z) = L(Z, Y), \quad L(Y, Z) = L(JY, JZ), \quad L(Y, JZ) = -L(JY, Z), \quad (7)$$

Ric and ρ are the Ricci tensor and scalar curvature of \mathcal{W} .

Let $x \in \mathcal{W}$ and $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_x \mathcal{W}$ and $\{e_{n+1}, \dots, e_{2m}\}$ be the orthonormal basis of $T^\perp \mathcal{W}$. We denote by \mathcal{H} , the mean curvature vector at x , that is

$$\mathcal{H}(x) = \frac{1}{n} \sum_{i=1}^n \omega(e_i, e_i), \quad (8)$$

Also, we set

$$\omega_{ij}^r = g(\omega(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, \quad r \in \{n+1, \dots, 2m\}$$

and

$$\|\omega\|^2 = \sum_{i,j=1}^n (\omega(e_i, e_j), \omega(e_i, e_j)). \quad (9)$$

For any $x \in \mathcal{W}$ and $X \in T_x \mathcal{W}$, we put $JX = TX + FX$, where TX and FX are the tangential and normal components of JX , respectively.

We denote by

$$\|T\|^2 = \sum_{i,j=1}^n g^2(Te_i, e_j).$$

Let \mathcal{W} be a Riemannian manifold. Denote by $\mathcal{K}(\pi)$ the sectional curvature of \mathcal{W} of the plane section $\pi \subset T_x \mathcal{W}, x \in \mathcal{W}$. The scalar curvature ρ for an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of the tangent space $T_x \mathcal{W}$ at x is defined by

$$\rho(x) = \sum_{i < j} K(e_i \wedge e_j).$$

Lemma 2.1. [6] Let $n \geq 2$ and x_1, x_2, \dots, x_n, b be real numbers such that

$$\left(\sum_{i=1}^n x_i\right)^2 = (n-1)\left(\sum_{i=1}^n x_i^2 + b\right)$$

then $2x_1x_2 \geq b$, with equality holds if and only if

$$x_1 + x_2 = x_3 = \dots = x_n.$$

In [1] A. Bejancu introduced the notion of CR-submanifolds, which is the generalization of invariant and anti-invariant submanifolds. In [4] B. Y. Chen introduced the notion of slant submanifolds as a generalization of CR-submanifolds.

Definition 2.1. A submanifold \mathcal{W} of a Bochner Kaehler manifold $\overline{\mathcal{W}}$ is said to be a slant submanifold if for any $x \in \mathcal{W}$ and $X \in T_x \mathcal{W}$, the angle between JX and $T_x \mathcal{W}$ is constant, i.e., the angle does not depend on the choice of $x \in \mathcal{W}$ and $X \in T_x \mathcal{W}$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of \mathcal{W} in $\overline{\mathcal{W}}$.

Invariant and anti-invariant submanifolds are the slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively and when $0 < \theta < \frac{\pi}{2}$, then slant submanifold is called proper slant submanifold.

Definition 2.2. Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f , a positive differentiable function on N_1 . The warped product of N_1 and N_2 is the Riemannian manifold $M = N_1 \times N_2 = (N_1 \times N_2, g)$, where $g = g_1 + f^2 g_2$

Definition 2.3. A Riemannian manifold \mathcal{W} is said to be Einstein manifold if the Ricci tensor is proportional to the metric tensor, that is, $Ric(X, Y) = \lambda g(X, Y)$ for some constant λ .

3. B. Y. Chen inequalities

In this section, we obtain B. Y. Chen inequalities for submanifolds of a Bochner Kaehler manifolds.

First we have,

Theorem 3.1. *Let \mathcal{W} be a submanifold of a Bochner Kaehler manifold $\overline{\mathcal{W}}$. Then, for each point $x \in \mathcal{W}$ and each plane section $\pi \subset T_x \mathcal{W}$, we have*

$$\mathcal{K}(\pi) \geq \left(\frac{5n^2 + 31n + 26 + 3\|T\|^2}{2(2n+2)(2n+4)} \right) \rho - \frac{n^2(n-2)}{2(n-1)} \|\mathcal{H}\|^2 - \frac{6}{2(2n+4)} Ric(e_i, Je_j)g(e_i, Je_j). \quad (10)$$

Equality holds if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_x \mathcal{W}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, \dots, e_{2m}\}$ of $T^\perp \mathcal{W}$ such that the shape operators takes the following forms

$$B_{n+1} = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 \\ 0 & 0 & \xi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi \end{pmatrix}, \alpha + \beta = \xi \quad (11)$$

and

$$B_r = \begin{pmatrix} \omega_{11}^r & \omega_{12}^r & 0 & \cdots & 0 \\ \omega_{12}^r & -\omega_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, r = n+2, \dots, 2m. \quad (12)$$

Proof. Using Gauss equation, the Riemannian curvature tensor of \mathcal{W} is given by

$$\begin{aligned} R(X, Y, Z, W) &= L(Y, Z)g(X, W) - L(X, Z)g(Y, W) + L(X, W)g(Y, Z) \\ &\quad - L(Y, W)g(X, Z) + M(Y, Z)g(JX, W) - M(X, Z)g(JY, W) \\ &\quad - M(X, W)g(JY, Z) - M(Y, W)g(JX, Z) - 2M(X, Y)(JZ, W) \\ &\quad - 2M(Z, W)g(JX, Y) + g(\omega(X, W), \omega(Y, Z)) - g(\omega(X, Z), \omega(Y, W)) \end{aligned}$$

for any $X, Y, Z, W \in T\mathcal{W}$.

$$\begin{aligned} \sum_{i,j} R(e_i, e_j, e_j, e_i) &= L(e_j, e_j)g(e_i, e_i) - L(e_i, e_j)g(e_j, e_i) + L(e_i, e_i)g(e_j, e_j) \\ &\quad - L(e_j, e_i)g(e_i, e_j) + M(e_j, e_j)g(Je_i, e_i) - M(e_i, e_j)g(Je_j, e_i) \\ &\quad - M(e_i, e_i)g(Je_j, e_j) - M(e_j, e_i)g(Je_i, e_j) - 2M(e_i, e_j)(Je_j, e_i) \\ &\quad - 2M(e_j, e_i)g(Je_i, e_j) + g(\omega(e_i, e_i), \omega(e_j, e_j)) - g(\omega(e_i, e_j), \omega(e_j, e_i)) \end{aligned}$$

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$$\begin{aligned}
&= L(e_j, e_j)g(e_i, e_i) - L(e_i, e_j)g(e_j, e_i) + L(e_i, e_i)g(e_j, e_j) \\
&\quad - L(e_j, e_i)g(e_i, e_j) - L(e_j, Je_j)g(Je_i, e_i) + L(e_i, Je_j)g(Je_j, e_i) \\
&\quad + L(e_i, Je_i)g(Je_j, e_j) + L(e_j, Je_i)g(Je_i, e_j) + 2L(e_i, Je_j)(Je_j, e_i) \\
&\quad + 2L(e_j, Je_i)g(Je_i, e_j) + g(\omega(e_i, e_i), \omega(e_j, e_j)) - g(\omega(e_i, e_j), \omega(e_j, e_i)).
\end{aligned} \tag{13}$$

Using (7), (8) and (9) in (13), we have

$$\begin{aligned}
\sum_{i,j} R(e_i, e_j, e_j, e_i) &= 2nL(e_i, e_i) - 2L(e_i, e_j)g(e_i, e_j) + 6L(e_i, Je_j)g(e_i, Je_j) \\
&\quad + n^2\|\mathcal{H}\|^2 - \|\omega\|^2.
\end{aligned}$$

Which simplifies to,

$$2\rho = 2(n-1)L(e_i, e_i) + 6L(e_i, Je_j)g(e_i, Je_j) + n^2\|\mathcal{H}\|^2 - \|\omega\|^2. \tag{14}$$

Combining (5) and (14), we have

$$\begin{aligned}
2\rho &= \frac{2(n-1)}{2n+4}Ric(e_i, e_i) - \frac{2(n-1)\rho}{2(2n+2)(2n+4)}g(e_i, e_i) \\
&\quad + \frac{6}{2n+4}Ric(e_i, Je_j)g(e_i, Je_j) - \frac{6\rho}{2(2n+2)(2n+4)}g(e_i, Je_j)g(e_i, Je_j) \\
&\quad + n^2\|\mathcal{H}\|^2 - \|\omega\|^2.
\end{aligned}$$

or

$$\begin{aligned}
2\rho &= \frac{6n^2 + 2n - 8 - 6\|T\|^2}{2(2n+2)(2n+4)}\rho + \frac{6}{2n+4}Ric(e_i, Je_j)g(e_i, Je_j) \\
&\quad + n^2\|\mathcal{H}\|^2 - \|\omega\|^2.
\end{aligned}$$

or

$$\left(2 - \frac{6n^2 + 2n - 8 - 6\|T\|^2}{2(2n+2)(2n+4)}\right)\rho = \frac{6}{2n+4}Ric(e_i, Je_j)g(e_i, Je_j) + n^2\|\mathcal{H}\|^2 - \|\omega\|^2.$$

Denoting by

$$\varepsilon = \left(2 - \frac{6n^2 + 2n - 8 - 6\|T\|^2}{2(2n+2)(2n+4)}\right)\rho - \frac{n^2(n-2)}{n-1}\|\mathcal{H}\|^2 - \frac{6}{2n+4}Ric(e_i, Je_j)g(e_i, Je_j),$$

we obtain

$$\varepsilon = n^2\|\mathcal{H}\|^2 - \|\omega\|^2 - \frac{n^2(n-2)}{n-1}\|\mathcal{H}\|^2.$$

or

$$n^2\|\mathcal{H}\|^2 = (n-1)(\varepsilon + \|\omega\|^2). \tag{15}$$

For chosen orthonormal basis, the above equation takes the form

$$\left(\sum_{i=1}^n \omega_{ii}^{n+1}\right)^2 = (n-1) \left[\sum_{i=1}^n (\omega_{ii}^{n+1})^2 + \sum_{i \neq j} (\omega_{ij}^{n+1})^2 + \sum_{r=n+1}^{2m} \sum_{i,j=1}^n (\omega_{ij}^r)^2 + \varepsilon \right]. \quad (16)$$

Using lemma 1 in (16), we have

$$2\omega_{11}^{n+1}\omega_{22}^{n+1} \geq \sum_{i \neq j} (\omega_{ij}^{n+1})^2 + \sum_{r=n+1}^{2m} \sum_{i,j=1}^n (\omega_{ij}^r)^2 + \varepsilon. \quad (17)$$

On the other hand, from Gauss equation we obtain

$$\mathcal{K}(\pi) = L(e_2, e_2) + L(e_1, e_1) + g(\omega(e_1, e_1), \omega(e_2, e_2) - g(\omega(e_1, e_2), \omega(e_2, e_1))). \quad (18)$$

Combing (5) and (18), we derive

$$\mathcal{K}(\pi) = \frac{4n+3}{(2n+2)(2n+4)}\rho + \omega_{11}^{n+1}\omega_{22}^{n+1} + \sum_{r=n+2}^{2m} \omega_{11}^r \omega_{22}^r - \sum_{r=n+1}^{2m} (\omega_{12}^r)^2. \quad (19)$$

Incorporating (17) in (19), we arrive at the inequality

$$\begin{aligned} \mathcal{K}(\pi) &\geq \frac{1}{2} \sum_{i \neq j} (\omega_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+1}^{2m} \sum_{i,j=1}^n (\omega_{ij}^r)^2 + \frac{1}{2} \varepsilon \\ &\quad + \frac{4n+3}{(2n+2)(2n+4)}\rho + \sum_{r=n+2}^{2m} \omega_{11}^r \omega_{22}^r - \sum_{r=n+1}^{2m} (\omega_{12}^r)^2. \end{aligned}$$

Which implies that

$$\mathcal{K}(\pi) \geq \frac{4n+3}{(2n+2)(2n+4)}\rho + \frac{1}{2}\varepsilon.$$

or

$$\mathcal{K}(\pi) \geq \left(\frac{5n^2 + 31n + 26 + 3\|T\|^2}{2(2n+2)(2n+4)} \right) \rho - \frac{n^2(n-2)}{2(n-1)} \|\mathcal{H}\|^2 - \frac{6}{2(2n+4)} \text{Ric}(e_i, Je_j) g(e_i, Je_j). \quad (20)$$

If the equality in (10) at a point p holds, then the inequality (20) become equality. In this case, we have

$$\begin{cases} \omega_{1j}^{n+1} = \omega_{2j}^{n+1} = \omega_{ij}^{n+1} = 0, & i \neq j > 2, \\ \omega_{ij}^r = 0, \forall i \neq j, & i, j = 3, \dots, 2m, \quad r = n+1, \dots, 2m, \\ \omega_{11}^r + \omega_{22}^r = 0, \forall r = n+2, \dots, 2m, \\ \omega_{11}^{n+2} + \omega_{22}^{n+1} = \dots = \omega_{11}^m + \omega_{22}^m = 0. \end{cases}$$

Now, if we choose e_1, e_2 such that $\omega_{12}^{n+1} = 0$ and we denote by $\alpha = \omega_{11}^r, \beta = \omega_{22}^r, \xi = \omega_{33}^{n+1} = \dots = \omega_{33}^r$. Therefore by choosing the suitable orthonormal basis the shape operators take the desired forms. \square

We conclude the following corollary from this theorem.

Corollary 3.2. *Let \mathcal{W} be a submanifold of a Bochner Kaehler manifold $\overline{\mathcal{W}}$ which is Einstein. Then, for each point $x \in \mathcal{W}$ and each plane section $\pi \subset T_x \mathcal{W}$, we have*

$$\mathcal{K}(\pi) \geq \left(\frac{5n^2 + 31n + 26 + 3\|T\|^2}{2(2n+2)(2n+4)} \right) \rho - \frac{n^2(n-2)}{2(n-1)} \|\mathcal{H}\|^2 - \frac{6\lambda}{2(2n+4)} \|T\|^2.$$

The equality at a point $x \in \mathcal{W}$ holds iff there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_x \mathcal{W}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, \dots, e_{2m}\}$ of $T^\perp \mathcal{W}$ such that shape operators of \mathcal{W} in $\overline{\mathcal{W}}$ at x have the forms (11) and (12).

Similarly, in case if \mathcal{W} is a slant submanifold of a Bochner Kaehler manifold $\overline{\mathcal{W}}$. We have the following theorem

Theorem 3.3. *Let \mathcal{W} be a slant submanifold of a Bochner Kaehler manifold $\overline{\mathcal{W}}$. Then, for each point $x \in \mathcal{W}$ and each plane section $\pi \subset T_x \mathcal{W}$, we have*

$$\mathcal{K}(\pi) \geq \left(\frac{5n^2 + 31n + 26 + 3\cos^2 \theta}{2(2n+2)(2n+4)} \right) \rho - \frac{n^2(n-2)}{2(n-1)} \|\mathcal{H}\|^2 - \frac{6}{2(2n+4)} \text{Ric}(e_i, J e_j) \cos \theta.$$

Equality holds if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_x \mathcal{W}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, \dots, e_{2m}\}$ of $T^\perp \mathcal{W}$ such that the shape operator takes the following forms

$$B_{n+1} = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 \\ 0 & 0 & \xi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi \end{pmatrix}, \alpha + \beta = \xi \quad (21)$$

and

$$B_r = \begin{pmatrix} \omega_{11}^r & \omega_{12}^r & 0 & \cdots & 0 \\ \omega_{12}^r & -\omega_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, r = n+2, \dots, 2m. \quad (22)$$

From this theorem, following corollaries can be easily deduced.

Corollary 3.4. *Let \mathcal{W} be a slant submanifold of a Bochner Kaehler manifold $\overline{\mathcal{W}}$, which is Einstein. Then, for each point $x \in \mathcal{W}$ and each plane section $\pi \subset T_x \mathcal{W}$, we have*

$$\mathcal{K}(\pi) \geq \left(\frac{5n^2 + 31n + 26 + 3\cos^2 \theta}{2(2n+2)(2n+4)} \right) \rho - \frac{n^2(n-2)}{2(n-1)} \|\mathcal{H}\|^2 - \frac{6\lambda}{2(2n+4)} \cos^2 \theta.$$

The equality holds at a point $x \in \mathcal{W}$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_x \mathcal{W}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, \dots, e_{2m}\}$ of $T^\perp \mathcal{W}$ such that shape operators of \mathcal{W} in $\overline{\mathcal{W}}$ at x have the forms (21) and (22).

Corollary 3.5. Let \mathcal{W} be a invariant submanifold of a Bochner Kaehler manifold $\overline{\mathcal{W}}$. Then, for each point $x \in \mathcal{W}$ and each plane section $\pi \subset T_x \mathcal{W}$, we have

$$\mathcal{K}(\pi) \geq \left(\frac{5n^2 + 31n + 26 + 3}{2(2n+2)(2n+4)} \right) \rho - \frac{n^2(n-2)}{2(n-1)} \|\mathcal{H}\|^2 - \frac{6}{2(2n+4)} \text{Ric}(e_i, J e_j).$$

The equality at a point $x \in \mathcal{W}$ holds iff there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_x \mathcal{W}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, \dots, e_{2m}\}$ of $T^\perp \mathcal{W}$ such that shape operators of \mathcal{W} in $\overline{\mathcal{W}}$ at x have the forms (21) and (22).

Corollary 3.6. Let \mathcal{W} be a anti-invariant submanifold of a Bochner Kaehler manifold $\overline{\mathcal{W}}$. Then, for each point $x \in \mathcal{W}$ and each plane section $\pi \subset T_x \mathcal{W}$, we have

$$\mathcal{K}(\pi) \geq \left(\frac{5n^2 + 31n + 26}{2(2n+2)(2n+4)} \right) \rho - \frac{n^2(n-2)}{2(n-1)} \|\mathcal{H}\|^2.$$

The equality at a point $x \in \mathcal{W}$ holds iff there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_x \mathcal{W}$ and orthonormal basis $\{e_{n+1}, e_{n+2}, \dots, e_{2m}\}$ of $T^\perp \mathcal{W}$ such that shape operators of \mathcal{W} in $\overline{\mathcal{W}}$ at x have the forms (21) and (22).

4. Warped product of CR-submanifolds of Bochner Kaehler manifolds

Let $\mathcal{W} = \mathcal{W}_T \times_f \mathcal{W}_\perp$ be the warped product CR-submanifolds of Bochner Kaehler manifold $\overline{\mathcal{W}}$ such that the invariant distribution is $D = T\mathcal{W}_T$ and anti-invariant distribution is $D^\perp = T\mathcal{W}_\perp$, where $f : \mathcal{W}_T \rightarrow \mathbb{R}$. Then the metric g on \mathcal{W} is given by [5]

$$g(X, Y) = \langle \pi_* X, \pi_* Y \rangle + (f \circ \pi)^2 \langle \sigma_* X, \sigma_* Y \rangle$$

where π and σ are the projection maps from \mathcal{W} onto \mathcal{W}_T and \mathcal{W}_\perp respectively.

It is easy to see that

$$T\mathcal{W} = D \oplus D^\perp \quad \text{and} \quad T^\perp \mathcal{W} = JD^\perp \oplus \nu, \quad (23)$$

where ν is the orthogonal distribution to JD^\perp in the normal bundle $T^\perp \mathcal{W}$.

From (23), we can write

$$\omega(X, Y) = \omega_{JD^\perp}(X, Y) + \omega_\nu(X, Y)$$

Also for warped product submanifold \mathcal{W} of $\overline{\mathcal{W}}$, we have [5]

$$\nabla_X Z = X(\log f)Z = \frac{X(f)}{f}Z \quad (24)$$

for any vector fields $X \in D$ and $Z \in D^\perp$.

Further, we can decompose $(\bar{\nabla}_X J)Y$ into the tangential and normal components as under

$$(\bar{\nabla}_X J)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y \quad (25)$$

where $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ denotes the tangential and normal components of $(\bar{\nabla}_X J)Y$

First we prove the following lemma

Lemma 4.1. *Let $\mathcal{W} = \mathcal{W}_T \times_f \mathcal{W}_\perp$ be a CR-warped product submanifold of a Bochner Kaehler manifold $\bar{\mathcal{W}}$. Then we have*

$$\omega_{JD^\perp}(JX, Z) = J\mathcal{P}_Z JX + X(\log f)JZ$$

$$g(\mathcal{P}_Z JX, W) = g(\mathcal{Q}_Z X, JW)$$

and

$$g(\omega(JX, Z), J\omega(X, Z)) - \|\omega_v(X, Z)\|^2 = g(\mathcal{Q}_Z X, J\omega_v(X, Z))$$

for $X \in D$ and $Z \in D^\perp$.

Proof. From Gauss equation, we have

$$\nabla_Z JX + \omega(JX, Z) - J(\nabla_Z X) - J\omega(X, Z) = \mathcal{P}_Z X + \mathcal{Q}_Z X$$

Using (24), we infer

$$\omega(JX, Z) = \mathcal{P}_Z X + \mathcal{Q}_Z X + J[X(\log f)Z] + J\omega(X, Z) - JX(\log f)Z.$$

Replace X by JX , we get

$$-\omega(X, Z) = \mathcal{P}_Z JX + \mathcal{Q}_Z JX + JX(\log f)JZ + J\omega(JX, Z) + X(\log f)Z.$$

We can write the above equation as

$$-\omega(X, Z) = \mathcal{P}_Z JX + \mathcal{Q}_Z JX + JX(\log f)JZ + J\omega_{JD^\perp}(JX, Z) + J\omega_v(JX, Z) + X(\log f)Z. \quad (26)$$

On comparing the tangential components, we obtain

$$\mathcal{P}_Z JX + J\omega_{JD^\perp}(JX, Z) + X(\log f)Z = 0,$$

or

$$J\omega_{JD^\perp}(JX, Z) = -\mathcal{P}_Z JX - X(\log f)Z$$

which shows that

$$\omega_{JD^\perp}(JX, Z) = J\mathcal{P}_Z JX + X(\log f)JZ, \quad (27)$$

for $X \in D$ and $Z \in D^\perp$.

Again on comparing the normal components in (26), we have

$$-\omega(X, Z) = \mathcal{Q}_Z JX + JX(\log f)JZ + J\omega_v(JX, Z)$$

from which we conclude that

$$\omega(JX, Z) = \mathcal{Q}_Z X + X(\log f)JZ + J\omega_v(X, Z)$$

or

$$\omega(JX, Z) - J\omega_v(X, Z) = \mathcal{Q}_Z X + X(\log f)JZ \quad (28)$$

By taking the inner product (28) with JW , we get

$$g(\omega_{JD^\perp}(JX, Z), JW) = g(\mathcal{Q}_Z X, JW) + X(\log f)g(JZ, JW) \quad (29)$$

Further using (27) in (29), we have

$$g(\mathcal{P}_Z JX, W) + X(\log f)g(Z, W) = g(\mathcal{Q}_Z X, JW) + X(\log f)g(Z, W)$$

from which we conclude that

$$g(\mathcal{P}_Z JX, W) = g(\mathcal{Q}_Z X, JW) \quad (30)$$

Also, by taking the inner product of (28) with $J\omega(X, Z)$, we find

$$g(\omega(JX, Z), J\omega(X, Z)) - \|\omega_v(X, Z)\|^2 = g(\mathcal{Q}_Z X, J\omega_v(X, Z)).$$

□

Theorem 4.2. Let $\mathcal{W} = \mathcal{W}_T \times_f \mathcal{W}_\perp$ be a warped product CR-submanifolds of Bochner Kaehler manifold $\overline{\mathcal{W}}$ with $\mathcal{P}_{D^\perp} D \in D$, then the squared norm of second fundamental form of \mathcal{W} in $\overline{\mathcal{W}}$ satisfies the following inequality

$$\|\omega\|^2 \geq \|\mathcal{P}_{D^\perp} D\|^2 + q\|grad_D(\log f)\|^2.$$

Proof. Let $\{X_1, \dots, X_p, X_{p+1} = JX_1, \dots, X_{2p} = JX_p\}$ be a local orthonormal frame of vector fields on N_T and $\{Z_1, \dots, Z_q\}$ be a local orthonormal frame of vector fields on N_\perp , where $2p + q = n$. Then we have

$$\begin{aligned} \|\omega\|^2 &= \sum_{i,j=1}^{2p} g(\omega(X_i, X_j), \omega(X_i, X_j)) + \sum_{\alpha=1}^q \sum_{j=1}^{2p} g(\omega(X_i, Z_\alpha), \omega(X_i, Z_\alpha)) \\ &\quad + \sum_{\alpha,\beta=1}^q g(\omega(Z_\alpha, Z_\beta), \omega(Z_\alpha, Z_\beta)) \end{aligned}$$

from above equation we can say that

$$\|\omega\|^2 \geq \sum_{j=1}^{2p} \sum_{\alpha=1}^q g(\omega(X_i, Z_\alpha), \omega(X_i, Z_\alpha))$$

Now from (27), we have

$$\|\omega\|^2 \geq \sum_{j=1}^{2p} \sum_{\alpha=1}^q g(J\mathcal{P}_{Z_\alpha}X_i - JX_i(\log f)JZ_\alpha, J\mathcal{P}_{Z_\alpha}X_i - JX_i(\log f)JZ_\alpha)$$

In view of the assumption $\mathcal{P}_{D^\perp}D \in D$, we have

$$\begin{aligned} \|\omega\|^2 &\geq \sum_{j=1}^{2p} \sum_{\alpha=1}^q \left[g(J\mathcal{P}_{Z_\alpha}X_i, J\mathcal{P}_{Z_\alpha}X_i) + g(JX_i(\log f)JZ_\alpha, JX_i(\log f)JZ_\alpha) \right] \\ &= \sum_{j=1}^{2p} \sum_{\alpha=1}^q \left[g(\mathcal{P}_{Z_\alpha}X_i, \mathcal{P}_{Z_\alpha}X_i) + (JX_i(\log f))^2 g(Z_\alpha, Z_\alpha) \right] \\ &= \|\mathcal{P}_{D^\perp}D\|^2 + \sum_{j=1}^{2p} \|JX_i(\log f)\|^2 q \\ &= \|\mathcal{P}_{D^\perp}D\|^2 + q\|grad_D(\log f)\|^2 \end{aligned}$$

where $grad_D$ denotes the gradient of some function on the distribution D .

Thus we have

$$\|\omega\|^2 \geq \|\mathcal{P}_{D^\perp}D\|^2 + q\|grad_D(\log f)\|^2.$$

□

Theorem 4.3. Let $\mathcal{W} = \mathcal{W}_T \times_f \mathcal{W}_\perp$ be a compact orientable warped product CR-submanifold of Bochner Kaehler manifold $\overline{\mathcal{W}}$. If $\mathcal{P}_{D^\perp}D \in D$ and $B_{\nabla_{JX_i}^\perp JZ}X_i = B_{\nabla_{X_i}^\perp JZ}JX_i$, then we have $\rho \leq 0$, and the equality holds iff $grad_D(\log f) = 0$.

Proof. Let $X \in D, Z \in D^\perp$, then from (4), we have

$$\overline{R}(X, JX, Z, JZ) = -2M(X, JX)g(Z, Z) - 2M(Z, JZ)g(X, X) \quad (31)$$

Now Codazzi equation is

$$\begin{aligned} \left[\overline{R}(X, Y)Z \right]^\perp &= \left\{ \nabla_X^\perp \omega(Y, Z) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) \right\} \\ &\quad - \left\{ \nabla_Y^\perp \omega(X, Z) - \omega(\nabla_Y X, Z) - \omega(X, \nabla_Y Z) \right\} \end{aligned}$$

In view of the last equation we may write

$$\begin{aligned} \overline{R}(X, JX, Z, JZ) &= g(\nabla_X^\perp \omega(JX, Z) - \omega(\nabla_X JX, Z) - \omega(JX, \nabla_X Z), JZ) \\ &\quad - g(\nabla_{JX}^\perp \omega(X, Z) - \omega(\nabla_{JX} X, Z) - \omega(X, \nabla_{JX} Z), JZ) \end{aligned} \quad (32)$$

We now compute each term of (32). First we have

$$Xg(\omega(JX, Z), JZ) = g(\overline{\nabla}_X \omega(JX, Z), JZ) + g(\omega(JX, Z), \overline{\nabla}_X JZ) \quad (33)$$

Using Weingarten formula we have

$$g(\bar{\nabla}_X^\perp \omega(JX, Z), JZ) = Xg(\omega(JX, Z), JZ) - g(\omega(JX, Z), \bar{\nabla}_X JZ) \quad (34)$$

Now from (28)

$$\omega(JX, Z) - J\omega_v(X, Z) = \mathcal{Q}_Z X + X(\log f)JZ \quad (35)$$

Taking the inner product of (35) with JZ , we have

$$g(\omega(JX, Z), JZ) - g(J\omega_v(X, Z), JZ) = g(\mathcal{Q}_Z X, JZ) + X(\log f)g(JZ, JZ) \quad (36)$$

Combining (30) and (36), we get

$$g(\omega(JX, Z), JZ) = g(\mathcal{P}_Z JX, Z) + X(\log f)\|Z\|^2 \quad (37)$$

Moreover

$$g(\omega(JX, Z), JZ) = X(\log f)g(Z, Z)$$

Hence we have

$$\begin{aligned} Xg(\omega(JX, Z), JZ) &= X \left\{ X(\log f)g(Z, Z) \right\} \\ &= X(X(\log f))g(Z, Z) + 2X(\log f)g(Z, \nabla_X Z) \\ &= X(X(\log f))\|Z\|^2 + 2(X(\log f))^2\|Z\|^2 \\ &= \left\{ X(X(\log f)) + 2(X(\log f))^2 \right\} \|Z\|^2 \end{aligned} \quad (38)$$

From (34) and (38), we get

$$g(\nabla_X^\perp \omega(JX, Z), JZ) = \left\{ X(X(\log f)) + 2(X(\log f))^2 \right\} \|Z\|^2 - g(\omega(JX, Z), \bar{\nabla}_X JZ) \quad (39)$$

Replacing X by JX in the above equation, we find

$$-g(\nabla_{JX}^\perp \omega(X, Z), JZ) = \left\{ JX(JX(\log f)) + 2(JX(\log f))^2 \right\} \|Z\|^2 + g(\omega(X, Z), \bar{\nabla}_{JX} JZ) \quad (40)$$

Also using (27) and $\mathcal{P}_{D^\perp} D \in D$, we conclude that

$$g(\omega_{JD^\perp}(JX, \nabla_X Z), JZ) = g(X(\log f)J\nabla_X Z, JZ) = (X(\log f))^2 g(Z, Z) = (X(\log f))^2 \|Z\|^2 \quad (41)$$

Replacing X by JX in the above equation, we find

$$g(\omega_{JD^\perp}(X, \nabla_{JX} Z), JZ) = -(JX(\log f))^2 \|Z\|^2 \quad (42)$$

Again using (27), we get

$$\omega_{JD^\perp}(\nabla_{JX} X, Z) = J\mathcal{P}_Z \nabla_{JX} X - J\nabla_{JX} X(\log f)JZ$$

or

$$g(\omega_{JD^\perp}(\nabla_{JX}X), JZ) = g(\mathcal{P}_Z \nabla_{JX}X, Z) - J\nabla_{JX}X(\log f)\|Z\|^2$$

The above equation can be written as

$$g(\omega(\nabla_{JX}X), JZ) = g(\mathcal{P}_Z \nabla_{JX}X, Z) - J\nabla_{JX}X(\log f)\|Z\|^2$$

But N_T is totally geodesic in \bar{N} which implies that $\nabla_{JX}X \in D$. Hence $\mathcal{P}_Z J\nabla_{JX}X \in D$. This makes the first term in the above equation zero and hence we have

$$g(\omega(\nabla_{JX}X), JZ) = -J\nabla_{JX}X(\log f)\|Z\|^2 \quad (43)$$

Similarly on replacing X by JX in the above equation, we have

$$g(\omega(\nabla_X JX), JZ) = -J\nabla_X JX(\log f)\|Z\|^2$$

Using Gauss equation, the last equation simplifies to

$$g(\omega(\nabla_X JX), JZ) = \nabla_X X(\log f)g(Z, Z) + \nabla_{JX} JX(\log f)g(Z, Z) - J\nabla_{JX}X(\log f)g(Z, Z) \quad (44)$$

Putting (39)~(44) into (32), we get

$$\begin{aligned} \bar{R}(X, JX, Z, JZ) &= \left\{ X(X(\log f)) + 2(X(\log f))^2 \right\} \|Z\|^2 - g(\omega(JX, Z), \nabla_X^\perp JZ) \\ &\quad - \nabla_X X(\log f)\|Z\|^2 - \nabla_{JX} JX(\log f)\|Z\|^2 + J\nabla^{JX} X(\log f)\|Z\|^2 \\ &\quad - (X(\log f))^2 \|Z\|^2 + \left\{ JX(JX(\log f)) + 2(JX(\log f))^2 \right\} \|Z\|^2 \\ &\quad + g(\omega(X, Z), \nabla_{JX}^\perp JZ) - J\nabla_{JX}X(\log f)\|Z\|^2 \end{aligned} \quad (45)$$

From (31) and (45)

$$\begin{aligned} &-2M(X, JX)g(Z, Z) - 2M(Z, JZ)g(X, X) \\ &= \left\{ X(X(\log f)) + 2(X(\log f))^2 \right\} \|Z\|^2 \\ &\quad - g(\omega(JX, Z), \nabla_X^\perp JZ) - \nabla_X X(\log f)\|Z\|^2 - \nabla_{JX} JX(\log f)\|Z\|^2 \\ &\quad - (X(\log f))^2 \|Z\|^2 + \left\{ JX(JX(\log f)) + 2(JX(\log f))^2 \right\} \|Z\|^2 \\ &\quad + g(\omega(X, Z), \nabla_{JX}^\perp JZ) - (JX(\log f))^2 \|Z\|^2 \end{aligned}$$

Putting $X = X_i$ and taking summation from 1 to p , we drive

$$\begin{aligned} &-2\|Z\|^2 \sum_{i=1}^p pM(X_i, JX_i) - 2M(Z, JZ)p \\ &= \sum_{i=1}^p \left\{ X_i(X_i(\log f)) + JX_i(JX_i(\log f)) - \nabla_{X_i} X_i(\log f) - \nabla_{JX_i} JX_i(\log f) \right\} \|Z\|^2 \\ &\quad + \sum_{i=1}^p \left\{ (X_i(\log f))^2 + (JX_i(\log f))^2 \right\} \|Z\|^2 \\ &\quad + \sum_{i=1}^p p \left[g(\omega(X_i, Z), \nabla_{JX_i}^\perp JZ) - g(\omega(JX_i, Z), \nabla_{X_i}^\perp JZ) \right] \|Z\|^2 \end{aligned}$$

from which we have

$$\begin{aligned}
& -2\|Z\|^2 \sum_{i=1}^p pM(X_i, JX_i) - 2M(Z, JZ)p \\
& = \Delta_D(\log f)\|Z\|^2 + \|\text{grad}_D(\log f)\|^2\|Z\|^2 \\
& \quad + \sum_{i=1}^p p \left[g(\omega(X_i, Z), \nabla_{JX_i}^\perp JZ) - g(\omega(JX_i, Z), \nabla_{X_i}^\perp JZ) \right] \|Z\|^2
\end{aligned}$$

Using (5) and (6) in the last equation we arrive at

$$\begin{aligned}
& \frac{-1}{n+2} \sum_{i=1}^p \left[\|Z\|^2 \text{Ric}(X_i, X_i) + \|X_i\|^2 \text{Ric}(Z, Z) \right] + \frac{\rho \|X_i\|^2 \|Z\|^2}{2(n+1)(n+2)} \\
& \quad = \Delta_D(\log f)\|Z\|^2 + \|\text{grad}_D(\log f)\|^2\|Z\|^2 \\
& \quad + \sum_{i=1}^p \left[g(\omega(X_i, Z), \nabla_{JX_i}^\perp JZ) - g(\omega(JX_i, Z), \nabla_{X_i}^\perp JZ) \right] \|Z\|^2
\end{aligned}$$

from which we have

$$\begin{aligned}
& \frac{-1}{n+2} \left[\|Z\|^2 \sum_{i=1}^p \text{Ric}(X_i, X_i) + p \text{Ric}(Z, Z) \right] + \frac{\rho p \|Z\|^2}{2(n+1)(n+2)} \\
& \quad = \Delta_D(\log f)\|Z\|^2 + \|\text{grad}_D(\log f)\|^2\|Z\|^2 \\
& \quad + \sum_{i=1}^p \left[g(B_{\nabla_{JX_i}^\perp JZ} X_i, Z) - g(B_{\nabla_{X_i}^\perp JZ} JX_i, Z) \right] \|Z\|^2
\end{aligned}$$

Since by assumption, we have $B_{\nabla_{JX_i}^\perp JZ} X_i = B_{\nabla_{X_i}^\perp JZ} JX_i$, then (46) becomes

$$\begin{aligned}
& \frac{-1}{n+2} \left[\sum_{i=1}^p \text{Ric}(X_i, X_i) + \frac{p}{\|Z\|^2} \text{Ric}(Z, Z) \right] + \frac{p\rho \|Z\|^2}{2(n+1)(n+2)} \\
& \quad = \Delta_D(\log f) + \|\text{grad}_D(\log f)\|^2
\end{aligned}$$

Integrating both sides and using Green's equation, the last equation simplifies to

$$\begin{aligned}
& \frac{-1}{n+2} \int \left[\sum_{i=1}^p \text{Ric}(X_i, X_i) + \frac{p}{\|Z\|^2} \text{Ric}(Z, Z) \right] dv + \int \frac{p\rho \|Z\|^2}{2(n+1)(n+2)} dv \\
& \quad = \int \|\text{grad}_D(\log f)\|^2 dv \tag{46}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
& \frac{-1}{n+2} \int \left[\sum_{i=1}^p \text{Ric}(JX_i, JX_i) + \frac{p}{\|Z\|^2} \text{Ric}(Z, Z) \right] dv + \int \frac{p\rho \|Z\|^2}{2(n+1)(n+2)} dv \\
& \quad = \int \|\text{grad}_D(\log f)\|^2 dv \tag{47}
\end{aligned}$$

Adding (46) and (47), we find

$$\begin{aligned}
& \frac{-1}{n+2} \int \left[\sum_{i=1}^p \text{Ric}(X_i, X_i) + \sum_{i=1}^p \text{Ric}(JX_i, JX_i) + \frac{2p}{\|Z\|^2} \text{Ric}(Z, Z) \right] dv \\
& \quad + \int \frac{2p\rho \|Z\|^2}{2(n+1)(n+2)} dv = 2 \int \|\text{grad}_D(\log f)\|^2 dv
\end{aligned}$$

from which we have

$$\frac{-1}{n+2} \int \left[\rho_D + \frac{2p}{\|Z\|^2} \text{Ric}(Z, Z) \right] dv + \int \frac{p\rho\|Z\|^2}{(n+1)(n+2)} dv = 2 \int \|\text{grad}_D(\log f)\|^2 dv$$

where ρ_D is the scalar curvature of distribution D . Further replacing Z by Z_α and taking summation from 1 to q on both sides. As

$$q \int \|\text{grad}_D(\log f)\|^2 dv \geq 0$$

we conclude that

$$\frac{-1}{n+2} \int \left[q\rho_D + 2p\rho_{D^\perp} \right] dv + \int \frac{pq^2\rho}{(n+1)(n+2)} dv \geq 0$$

This shows that

$$\frac{pq^2}{(n+1)(n+2)} \int \rho dv \geq \frac{1}{n+2} \int \left[q\rho_D + 2p\rho_{D^\perp} \right] dv$$

or

$$\int \rho dv \geq (n+1) \int \left[\frac{\rho_D}{pq} + \frac{2(n+1)}{q^2} \rho_{D^\perp} \right] dv \quad (48)$$

Thus we have

$$\int \left[\rho_D + \rho_{D^\perp} \right] dv \geq \int \left[\frac{(n+1)}{pq} \rho_D + \frac{2(n+1)}{q^2} \rho_{D^\perp} \right] dv$$

From we have the following observations. Either $(n+1) \leq pq$ and $2(n+1) \leq q^2$ or $\rho_D \leq 0$ and $\rho_{D^\perp} \leq 0$ that id $\rho = \rho_D + \rho_{D^\perp} \leq 0$. Equality holds if and only if either $(n+1) = pq$ and $2(n+1) = q^2$ or $\text{grad}_D(\log f) = 0$. \square

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